

On the signature of pressure in gravity

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Abstract

When pressure is not negligible in comparison with energy density, the external gravitational field and the motion of particles in it are modified. For spherically symmetric body two effective mass parameters determine the external gravitational field and the motion of particles in it. For distances much larger then the gravitational radius we use the linearized Einstein equations to consider the effects of pressure on test particle motion.

1 Introduction

For strongly compressed star the pressure may noticeably influence the gravitational field not only inside the star but also outside if it [1] In the region $\frac{r_g}{r} \ll 1$ it is sufficient to use the linearized Einstein equation to consider the influence of pressure on gravitational field. This is done in Section 2. In Section 3 we approach the problem using Schwinger's Theory of Sources to make sure that it gives the same results as we obtain in this paper. In section 4 we consider the influence of pressure on the motion of test particle.

2 Pressure in gravity.

We assume that the energy-momentum tensor of the gravitating body is

$$T^{\mu\nu} = \text{diag}(\epsilon, p, p, p), \quad (1)$$

The linearized Einstein equation may be written in the form ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$)

$$\Delta h_{\mu\nu} = -\frac{16\pi G}{c^4} \bar{T}_{\mu\nu}, \quad \bar{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T^l_l, \quad \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1), \quad (2)$$

Here

$$\bar{T}_{00} = \frac{1}{2}(\epsilon + 3p), \quad \bar{T}_{11} = \bar{T}_{22} = \bar{T}_{33} = \frac{1}{2}(\epsilon - p). \quad (3)$$

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The solution of (2) is well known,

$$h_{\mu\nu}(\vec{x}) = 4\frac{G}{c^4} \int \frac{\bar{T}_{\mu\nu}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'. \quad (4)$$

For spherically symmetric body it follows from (4) that outside the body

$$h_{00} = \frac{2Gm_t}{rc^2}, \quad h_s \equiv h_{11} = \frac{2Gm_s}{rc^2}. \quad (5)$$

Here m_t is the effective mass (with the pressure taken into account) defining h_{00} and similarly for space part of metric $h_s \equiv h_{11}$:

$$m_t = c^{-2} \int [\epsilon + 3p] d^3x, \quad m_s = c^{-2} \int [\epsilon - p] d^3x. \quad (6)$$

The subscripts t and s remind us about time- and space parts of the metric. We also define

$$r_{gt} = \frac{2Gm_t}{c^2}, \quad r_{gs} = \frac{2Gm_s}{c^2}. \quad (7)$$

In spherical coordinate system we get

$$\begin{aligned} -ds^2 &= g_{00}(dx^0)^2 + g_s(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2), \\ g_{00} &= -(1 - h_{00}) = -(1 + \frac{2\varphi}{c^2}), \quad g_s \equiv g_{11} = 1 + h_s = 1 + \frac{2Gm_s}{rc^2}, \end{aligned} \quad (8)$$

see (5). The equation of motion for radial coordinate is

$$\frac{d^2r}{ds^2} = -\Gamma_{\mu\nu}^r \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}, \quad x^0 = ct. \quad (9)$$

For particle at rest

$$\frac{d^2r}{ds^2} = -\Gamma_{00}^r = \frac{1}{2} g^{rr} \frac{\partial g_{00}}{\partial r} = \frac{1}{2} \frac{\partial h_{00}}{\partial r}. \quad (10)$$

With the help of (5) we find

$$\frac{d^2r}{dt^2} = -\frac{\partial \varphi}{\partial r}, \quad \varphi = -\frac{Gm_t}{r}. \quad (11)$$

In contrast to Schwarzschild solution the parameter m_t depends on pressure. In view of this it is instructive to make a short excursion to Schwinger's Theory of Sources [2].

3 Pressure in Schwinger's Theory of Sources

We denote Schwinger's total energy-momentum tensor $T^{\mu\nu}$ in eq. (4.34) Ch.2 in [2] as $\theta^{\mu\nu}$ and write it in the form

$$E(x^0) = -\frac{G}{c^4} \int \frac{d^3x d^3x'}{|\vec{x} - \vec{x}'|} [\theta^{\mu\nu}(\vec{x}, x^0) \theta_{\mu\nu}(\vec{x}', x^0) - \frac{1}{2} \theta(\vec{x}, x^0) \theta(\vec{x}', x^0)], \quad \theta = \theta^\nu{}_\nu. \quad (12)$$

Using $\theta^{\mu\nu} = T^{\mu\nu} + t^{\mu\nu}$ we get from (12) the interaction energy

$$E_{int}(x^0) = -\frac{G}{c^4} \int \frac{d^3x d^3x'}{|\vec{x} - \vec{x}'|} \{ [T^{\mu\nu}(\vec{x}, x^0) t_{\mu\nu}(\vec{x}', x^0) + (\vec{x} \leftrightarrow \vec{x}')] - \frac{1}{2} [T(\vec{x}, x^0) t(\vec{x}', x^0) + (\vec{x} \leftrightarrow \vec{x}')] \}. \quad (13)$$

Here $(\vec{x} \leftrightarrow \vec{x}')$ means terms obtained from the preceding ones by substitution $(\vec{x} \leftrightarrow \vec{x}')$.

I. We first assume that

$$t^{\mu\nu}(\vec{x}, x^0) = m' c^2 \delta_{\mu 0} \delta_{\nu 0} \delta(\vec{x} - \vec{R}), \quad (14)$$

i.e. the test particle is at rest. Then, from (13) and (14) we find

$$E_{int} = -\frac{Gm'}{c^2} \int \frac{d^3x}{|\vec{x} - \vec{R}|} \{ T^{00}(\vec{x}) - \frac{1}{2} [T^{00} - T_{kk}] + (\vec{x} \leftrightarrow \vec{x}') \} = -\frac{Gm'}{c^2} \int \frac{d^3x}{|\vec{x} - \vec{R}|} \frac{1}{2} \{ [\epsilon(\vec{x}) + 3p(\vec{x})] + (\vec{x} \leftrightarrow \vec{x}') \} = -\frac{Gm'}{c^2} \int \frac{d^3x}{|\vec{x} - \vec{R}|} \{ [\epsilon(\vec{x}) + 3p(\vec{x})] \}. \quad (15)$$

For spherically symmetric gravitating body we get the Newtonian law with $m \rightarrow m_t$:

$$E_{int} = -\frac{Gm'm_t}{R} = m'\varphi \quad (16)$$

in agreement with (11).

II. Next, we consider the interaction of a photon beam:

$$t^{\mu\nu} = \sigma p^\mu p^\nu, \quad p^2 = -(p^0)^2 + \vec{p}^2 = 0, \quad (17)$$

see eq. (4.38) in Ch.2 in [2]. Using (17) and (13) we obtain

$$E_{int} = -\frac{G}{c^4} \int \frac{d^3x d^3x'}{|\vec{x} - \vec{x}'|} \sigma p^0 \{ [\epsilon(\vec{x}) + p(\vec{x})] + (\vec{x} \leftrightarrow \vec{x}') \} = -\frac{G}{c^4} \int \frac{d^3x d^3x'}{|\vec{x} - \vec{x}'|} \sigma 2p^0 \{ [\epsilon(\vec{x}) + p(\vec{x})] \}, \quad (18)$$

i.e. the interaction energy of the photon with the gravitating body is twice the Newtonian value (which is obtained by substituting $m'c^2 \rightarrow p^0, m \rightarrow m_{eff}$), cf. [2];

$$m_{eff} = \frac{m_t + m_s}{2}. \quad (19)$$

It follows from here that the deflexion angle for the photon flying through the gravitational field of a spherically symmetric body is

$$\theta = \frac{4Gm_{eff}}{\rho c^2} = \frac{2r_{eff}}{\rho}, \quad (20)$$

see eq. (4.41) in Ch2 in [2]. Here ρ is the impact parameter. The important thing for us is that $m_{eff} \neq m_t$.

4 Test particle in metric (8)

We use the Hamilton-Jacobi equation, see §101 in [3]

$$g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = -m'^2 c^2. \quad (21)$$

With

$$g^{00} = -(1 - \frac{r_{gt}}{r})^{-1}, \quad g_s^{-1} = (1 + \frac{r_{gt}}{r})^{-1}$$

we have

$$(1 - \frac{r_{gt}}{r})^{-1} (\partial S / \partial t)^2 - (1 + \frac{r_{gs}}{r})^{-1} [(\partial S / \partial r)^2 + r^{-2} (\partial S / \partial \theta)^2] = m'^2 c^2. \quad (22)$$

S has the form

$$S = -\mathcal{E}_0 t + M\theta + S_r, \quad (23)$$

where M is the angular momentum. Using this in (22), we find

$$S_r = \int \sqrt{F(r)} dr, \quad F(r) = \frac{\mathcal{E}_0^2}{c^2} \frac{1 + \frac{r_{gs}}{r}}{1 - \frac{r_{gt}}{r}} - \frac{M^2}{r^2} - m'^2 c^2 (1 + \frac{r_{gs}}{r}). \quad (24)$$

The function $r = r(t)$ is given by the condition $\frac{\partial S}{\partial \mathcal{E}_0} = \text{const}$ and the trajectory is obtained from $\frac{\partial S}{\partial M} = \text{const}$, see §101 in [3]:

$$t = \frac{\mathcal{E}_0}{c^2} \int \frac{1 + \frac{r_{gs}}{r}}{1 - \frac{r_{gt}}{r}} F^{-\frac{1}{2}} dr, \quad (25)$$

$$\theta = \int \frac{M}{r^2} F^{-\frac{1}{2}} dr, \quad (26)$$

In the nonrelativistic limit $c \rightarrow \infty$ we have

$$\frac{\mathcal{E}_0^2 - m'^2 c^4}{c^2} \approx E_{nonrel} 2m', .$$

$$F(r) \approx \frac{\mathcal{E}_0^2}{c^2} (1 + \frac{r_{gs} + r_{gt}}{r}) - \frac{M^2}{r^2} - m'^2 c^2 (1 + \frac{r_{gs}}{r}) \approx \quad (27)$$

$$2m' (E_{nonrel} - E_{int}) - \frac{M^2}{r^2}, \quad E_{int} = -\frac{Gm'm_t}{r}. \quad (28)$$

Using (27), (28) in eqs. (25) and (26) we get the equations for the nonrelativistic particle. This agrees with (11) and (16).

For $m' \rightarrow 0$ we see from (25), (26) that in the first approximation the role of m plays $m_{eff} = (m_s + m_t)/2$ in agreement with (19)

Starting from (26) with $m' = 0$ it is not difficult to obtain corrections to the leading term (20) in the form of powers of $1/\rho$.

In terms of

$$\delta = \frac{r_{gt}}{\rho}, \quad r_{gs} = \zeta r_{gt}, \quad (29)$$

where ρ is the impact parameter, we have instead (26)

$$\theta = \int \sqrt{\frac{1-u\delta}{f(u)}} du, \quad f(u) = 1 - u^2 + (u\zeta + u^3)\delta = (u - u_1)(u - u_2)(u - u_3)\delta. \quad (30)$$

For photon $M = \rho\mathcal{E}_0/c$. and θ at half of the trajectory is, cf. [4]

$$\theta_{\frac{1}{2}} = (u_3\delta)^{-\frac{1}{2}} \int_0^{u_2} \sqrt{\frac{1-u\delta}{1-uu_3^{-1}}} R(u)^{-1/2} du, \quad R(u) = (u_2 - u)(u - u_1). \quad (31)$$

This equation coincides with eq. (42) in [4] and $f(u)$ is the same as that in eq.(2) in [4]. So, using the results obtained there, it is easy to get the deflexion angle for photon

$$\theta = 2\left(\theta_{\frac{1}{2}} - \frac{\pi}{2}\right) = (1 + \zeta)\delta + \pi\frac{1 + \zeta}{2}\delta^2 + \dots. \quad (32)$$

The leading term

$$(1 + \zeta)\delta = \frac{r_{gt} + r_{gs}}{\rho} = \frac{4Gm_{eff}}{\rho c^2}$$

agrees with (20). Here eqs. (7), (19) and (29) were used.

5 Conclusion

When pressure becomes noticeable, two mass parameters m_t and m_s differ from one another. In this case the Birkhoff theorem can be valid only with accuracy of order $\frac{m_t - m_s}{m_t}$ because at the beginning of contraction $m_t = m_s = m$. In the considered approximation the presence of pressure increases the pull towards the gravitating body more for nonrelativistic particle then for ultrarelativistic one.

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References

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